Algebra-II
B. Math - First year
Semestral Exam
2012-2013

Time: 3hrs
Max score: 100

Answer question **1** and any **four** from the rest.

(1) State true or false. Justify your answers. No marks will be awarded in the absence of proper justification.
(a) The subgroup $SL_n(\mathbb{F}_p) = \{A \in GL_n(\mathbb{F}_p) : \det(A) = 1\}$ is of index p in $GL_n(\mathbb{F}_p)$.
(b) For an idempotent matrix A (ie. $A^2 = A$), rank is same as number of non-zero eigen values.
(c) The eigenvalues of a complex skew hermitian matrix A (ie. $A^* = -A$) are all real.
(d) The columns of an $n \times n$ unitary matrix form an orthonormal basis of $\mathbb{C}^n$. (5+5+5+5)

(2) Let V be a vector space over some field $\mathbb{F}$ and let $\dim V = n$. Let $T : V \longrightarrow V$ be a linear operator.
(a) Let λ be an eigenvalue of T. Then show that geometric multiplicity (ie. dimension of the eigenspace) of λ is less than or equal to its algebraic multiplicity.
(b) Show that T is diagonalizable if and only if for every eigenvalue λ of T, geometric multiplicity equals algebraic multiplicity. (12+8)

(3) (a) Define hermitian matrices. Show that the set of all $n \times n$ hermitian matrices form a vector space over the reals. Find a basis for this space and determine its dimension.
(b) Let W be the space of all real $n \times n$ matrices of trace zero. Find a subspace W' such that $\mathbb{R}^{n\times n} = W \oplus W'$. (12+8)

(4) (a) Show that a real $n \times n$ matrix A represents the dot product with respect to some basis of $\mathbb{R}^n$ iff $A = P^tP$ for some invertible matrix $P \in GL_n(\mathbb{R})$.
(b) Let V be the vector space of all real polynomials of degree ≤ 3. Define a bilinear form on V by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx.$$

Find an orthonormal basis for V. (10+10)

(5) Let the matrix of a symmetric bilinear form $\langle , \rangle$ on $\mathbb{R}^3$ be $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ with respect to the standard basis. Find an orthogonal basis of $\mathbb{R}^3$ with respect to $\langle , \rangle$. Hence find the signature of the form. (20)

(6) (a) Prove that a complex matrix M is normal if and only if there is a unitary matrix P such that PMP^* is diagonal.
(b) Hence show that every conjugacy class in the unitary group $U_n = \{P : P^*P = I_n\}$ contains a diagonal matrix. (16+4)